Lecture 08

1. Review

Prolongation of diffeomorphisms. A local diffromorphism $g : X \times U \to X \times U$ induces $\mathsf{pr}^n g : X \times U^{(n)} \to X \times U^{(n)}$, which is defined as follows. Take any function $u = f(x)$ such that $u^{(n)}(x) = f^{(n)}(x)$. Then transform this by g [1] to get new function $\tilde{u} = \tilde{f}(\tilde{x})$ for which we let $\tilde{f} =: g \circ f$. Now we define the image of $(x, u^{(n)})$ under $\mathsf{pr}^n g$ is $(\tilde{x}, \tilde{f}^{(n)}(\tilde{x}))$.
For V a vector field on $X \times U$, $\mathsf{pr}^n V$ is a vector field on $X \times U^{(n)}$ defined as follows. Let $g_\varepsilon := \exp(\varepsilon V)$ which is a local diffeomorphism of $X \times U$. Then $\mathsf{pr}^n g_\varepsilon(x, u^{(n)})$ is a curve in $X \times U^{(n)}$ parametrized by ε. We identify $\mathsf{pr}^n V$ with $\frac{d}{d\varepsilon}\big|_{\varepsilon=0} \mathsf{pr}^n g_\varepsilon(x, u^{(n)})$.

Symmetries of differential equations. Given a system of partial differential equations $\Delta(x, u^{(n)}) = 0$ with $\Delta = (\Delta_1, \ldots, \Delta_l)$ and $\mathcal{S}_\Delta$ the zero set of Δ, a symmetry g refers to a diffeomorphism $X \times U$ to itself such that $\mathsf{pr}^n g : \mathcal{S}_\Delta \to \mathcal{S}_\Delta$.

Remark 1.1. Note this definition is different from saying $\mathsf{pr}^n g$ sends a solution to another solution. For a certain class of differential equations has the solution at every point of $\mathcal{S}_\Delta$. But some differential equation e.g. over-determined system of differential equation has the solution only at some points of $\mathcal{S}_\Delta$.

If g is such a diffeomorphism and if $u = f(x)$ is a solution of $\Delta = 0$ then $\tilde{u} = (g \circ f)(\tilde{x})$ is also a solution of $\Delta = 0$. In fact, let f be a solution then $(x, f^{(n)}(x)) \in \mathcal{S}_\Delta$ for all $x \in X$. Now $(\tilde{x}, \tilde{f}^{(n)}(\tilde{x})) = \mathsf{pr}^n g(x, f^{(n)}(x)) \in \mathcal{S}_\Delta$. Hence $\tilde{u} = \tilde{f}(\tilde{x})$ is a solution. This justifies why we call g a *symmetry* of $\Delta = 0$.

2. Infinitesimal symmetries

Infinitesimal symmetries. We look for vector fields V on $X \times U$ such that $\mathsf{pr}^n V$ is tangent to $\mathcal{S}_\Delta \subset X \times U^{(n)}$. Such V is called an *infinitesimal symmetry* of $\Delta = 0$.

Remark 2.1. The space of infinitesimal symmetries is essentially equivalent to the local symmetric group of identity component. If one is finite dimensional so is the other and vice versa. But the infinitesimal symmetries are preferred since it admits concrete calculation many times.

Note $(\mathsf{pr}^n V)\Delta = 0$ on $\Delta = 0$ if and only if $(\mathsf{pr}^n V)\Delta = \sum_{i=1}^q Q_i \cdot \Delta_i$ for some differential function $Q_i(x, u^{(n)})$.

Properties of pr^n*: prolongation of vector fields.* Let $\mathcal{X}(\cdot)$ denote the space of smooth vector fields defined locally on the given manifold. pr^n is a map $\mathcal{X}(X \times U) \to \mathcal{X}(X \times U^{(n)})$ such that

(1) $\mathsf{pr}^n(aV + bW) = a\mathsf{pr}^n V + b\mathsf{pr}^n W$
(2) $\mathsf{pr}^n[V, W] = [\mathsf{pr}^n V, \mathsf{pr}^n W]$

for constants a, b and local smooth vector fields V, W. Hence pr^n is a Lie algebra homomorphism.

Example 2.2. Let $p = q = 1$ and $u(x)$ solve $\Delta(x, u, u_x) = (u - x)u_x + u + x = 0$. Show $SO(2)$ is a symmetry group.

Solution. Symmetric group itself is difficult to find by calculation whereas infinitesimal generators thereof are *more calculable*. The reason is that they are solved

[1] We consider the case g is close enough to identity mapping.

from the linearized differential equations. Once the generator V found, $\exp(\varepsilon V)$ gives the symmetric group element. In the viewpoint above, we are well enough to show $V = -u\frac{\partial}{\partial x} + x\frac{\partial}{\partial u}$, the generator of $SO(2)$ is an infinitesimal symmetry of $\Delta = 0$. Recalling $\mathsf{pr}^1 V = -u\frac{\partial}{\partial x} + x\frac{\partial}{\partial u} + (1+u_x^2)\frac{\partial}{\partial u_x}$, we have

$$\begin{aligned}(\mathsf{pr}^1 V)\Delta &= -u(-u_x+1) + x(u_x+1) + (1+u_x^2)(u-x)\\ &= u_x \cdot (u + x + uu_x - xu_x)\\ &= u_x \cdot \Delta\end{aligned}$$

which is 0 on $\Delta = 0$. ||

This is how this example was found. An ordinary differential equation

$$\frac{dr}{d\theta} = r$$

in polar co-ordinates (r, θ) has the solution $r(\theta) = r(0)\exp(\theta)$ whose graph spirals out. This obviously solves the same ordinary differential equation after *rotation* and so $SO(2)$ is a symmetric group of this ordinary differential equation. We convert this into $(x, u) \in \mathbb{R}^2$ co-ordinates by $x = r\cos\theta$ and $u = r\sin\theta$ to get $(u-x)u_x + u + x = 0$.

3. Prolongation formula for vector fields

THEOREM 3.1. *Let*

$$V = \sum_{i=1}^{p} \xi^i(x,u)\frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \phi_\alpha(x,u)\frac{\partial}{\partial u^\alpha}$$

be a vector field on $X \times U$. Then

$$\mathsf{pr}^n V = V + \sum_{\alpha=1}^{q}\sum_{|J|\le n} \phi_\alpha^J(x, u^{(n)})\frac{\partial}{\partial u_J^\alpha},$$

where

$$\phi_\alpha^{J,k} = D_k(\phi_\alpha^J) - \sum_{i=1}^{p}(D_k\xi^i)u_{Ji}^\alpha.$$

The multi index $J = (j_1 \dots j_m)$, $j_i = 1 \dots p$ is understood as unordered, $|J| := m$ and $Jk := (j_1 \dots j_m\ k)$, $j_i, k = 1 \dots p$.

EXAMPLE 3.2. We verify this formula by the previous example. For $p = q = 1$, let $V = -u\frac{\partial}{\partial x} + x\frac{\partial}{\partial u}$ and $\mathsf{pr}^1 V = V + \phi^x\frac{\partial}{\partial u_x}$. According to the formula above, $\phi^x = D_x\phi - ((D_x\xi)u_x = 1 - (-u_x)u_x = 1 + u_x^2$ the same result as before.

3.1. Derivation of formula for special vector fields.

(1). Let $V = \sum_{i=1}^{p}\xi^i(x)\frac{\partial}{\partial x^i}$ with $q = 1$ and $\exp(\varepsilon V) := g_\varepsilon$. Let $g_\varepsilon \cdot (x,u) = (\tilde{x}, \tilde{u}) = (\Xi_\varepsilon(x), u)$ then $\frac{\partial}{\partial\varepsilon}\Big|_{\varepsilon=0}\Xi^i = \xi^i(x)$. Put $(\mathsf{pr}^1 g_\varepsilon)(x, u^{(1)}) = (\tilde{x}, \tilde{u}^{(1)}) = (\Xi_\varepsilon(x), u, \tilde{u}_j)$ where $\tilde{u}_j$ is to find. Let $u = f(x)$ be any function that fits $(x, u^{(1)})$ and let $\tilde{f}_\varepsilon = g_\varepsilon \cdot f$ be the transformed function which is given by $\tilde{u} = \tilde{f}_\varepsilon(\tilde{x}) =$

$f(\Xi_\varepsilon^{-1}(\tilde{x})) = f(\ \Xi_{-\varepsilon}(\tilde{x})$. Then $\tilde{u}_j = \frac{\partial \tilde{f}_\varepsilon}{\partial \tilde{x^j}}(\tilde{x}) = \frac{\partial f}{\partial x^k}(\Xi_{-\varepsilon}(\tilde{x}))\cdot \frac{\partial\ \Xi^k_{-\varepsilon}}{\partial \tilde{x}_j}(\tilde{x}) = \sum_{k=1}^p \frac{\partial\ \Xi^k_{-\varepsilon}}{\partial \tilde{x^j}}(\tilde{x})\cdot u_k$ and so $\mathsf{pr}^1 g_\varepsilon(x, u^{(1)}) = (\ \Xi_\varepsilon(x), u, \sum_{k=1}^p \frac{\partial\ \Xi^k_{-\varepsilon}}{\partial \tilde{x^j}}(\tilde{x})\cdot u_k)$.

$$\begin{aligned}\mathsf{pr}^1 V &= \sum_{i=1}^p \xi^i \frac{\partial}{\partial x^i} + 0\cdot\frac{\partial}{\partial u} + \sum_{k=1}^p \frac{\partial}{\partial \tilde{x^j}}\left(\frac{\partial}{\partial\varepsilon}\Xi^k_{-\varepsilon}(\tilde{x})\right)\cdot u_k \\ &= \sum_{i=1}^p \xi^i \frac{\partial}{\partial x^i} + \sum_{k=1}^p \left[-\frac{\partial \xi^k}{\partial \tilde{x^j}}(\tilde{x}) + \sum_{i=1}^p \frac{\partial^2\ \Xi^k_{-\varepsilon}}{\partial \tilde{x}^j \partial \tilde{x}^i}\xi^i\right]\cdot u_k\Bigg|_{\varepsilon=0} \\ &= \sum_{i=1}^p \xi^i \frac{\partial}{\partial x^i} + \sum_{k=1}^p \left(-\frac{\partial \xi^k}{\partial x^j}(x)\right)\cdot u_k.\end{aligned}$$

Therefore

$$\mathsf{pr}^1 V = V + \sum_{j=1}^p \left(-\sum_{k=1}^p \frac{\partial \xi^k}{\partial x^j}(x)\cdot u_k\right)\cdot \frac{\partial}{\partial u^j}$$

where we put $\phi^j = -\sum_{k=1}^p \frac{\partial \xi^k}{\partial x^j}(x)\cdot u_k$.

(2). Let $V = \phi(x,u)\frac{\partial}{\partial u}$ with $q = 1$ and $\exp(\varepsilon V) =: g_\varepsilon$. Set $g_\varepsilon(x,u) = (x, \Phi_\varepsilon(x,u))$. Then $\frac{\partial \Phi}{\partial \varepsilon}\mid_{\varepsilon=0} = \phi(x,u)$. For any function $u = f(x)$ let $\tilde{f}_\varepsilon := g_\varepsilon \cdot f$ then $\tilde{u} = \tilde{f}(\tilde{x}) = \Phi_\varepsilon(x, f(x))$ and $\tilde{u}_j = \frac{\partial \tilde{u}}{\partial \tilde{x^j}} = \frac{\partial \tilde{u}}{\partial x^j} = \frac{\partial \Phi_\varepsilon}{\partial x^j} + \frac{\partial \Phi_\varepsilon}{\partial u}\frac{\partial f}{\partial x^j}$. So, $g_\varepsilon(x, u, u^j, 1\le j\le p) = (x, \Phi_\varepsilon(x,u), \frac{\partial \Phi_\varepsilon}{\partial x^j} + \frac{\partial \Phi_\varepsilon}{\partial u}u_j, 1 \le j \le p)$, which we differentiate in ε and evaluate at $\varepsilon = 0$ to get

$$\begin{aligned}\mathsf{pr}^1 V &= \left(0, \phi(x,u), \frac{\partial \phi}{\partial x^j} + \frac{\partial \phi}{\partial u}\cdot u_j\right) \\ &= \phi\frac{\partial}{\partial u} + \sum_{j=1}^p \left(\frac{\partial \phi}{\partial x^j} + \frac{\partial \phi}{\partial u}\cdot u_j\right)\frac{\partial}{\partial u_j} \\ &= \phi\frac{\partial}{\partial u} + \sum_{j=1}^p \phi^j \frac{\partial}{\partial u_j}\end{aligned}$$

where $\phi^j := \frac{\partial \phi}{\partial x^j} + \frac{\partial \phi}{\partial u_j}$.